ARO 17074.28-MA

AD A134 627

ARO Report 82-2

PROCEEDINGS OF THE TWENTY-SEVENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING

Sponsored by

The Army Mathematics Steering Committee

on Behalf of

THE CHIEF OF RESEARCH, DEVELOPMENT AND ACQUISITION

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 17074.28-MA	2. GOVT ACCESSION NO. AD-A134 ??7 N/A	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) The U. S. Army (BRL'S) Kinetic Energy Penetrator Problem: Estimating the Probability of Response for a Given Stimulus		5. TYPE OF REPORT & PERIOD COVERED Reprint
		6. PERFORMING ORG. REPORT NUMBER N/A
7. AUTHOR(s) Thomas A. Mazzuchi Nozer D. Singpurwalla		8. CONTRACT OR GRANT NUMBER(s) DAAG29 80 C 0067
9. PERFORMING ORGANIZATION NAME AND ADDRESS George Washington University Washington, DC		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE 1982
		13. NUMBER OF PAGES 33
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Reliability and Risk Analysis

Abstract
of
Serial GWU/IRRA/TR-81/4
21 December 1981

THE U.S. ARMY (BRL'S) KINETIC ENERGY PENETRATOR PROBLEM: ESTIMATING THE PROBABILITY OF RESPONSE FOR A GIVEN STIMULUS

Thomas A. Mazzuchi
Nozer D. Singpurwalla

The crew compartment of an army vehicle is protected by an armor plate. It is desired to test the strength of this armor plate in order to assess its appropriateness for use in the vehicle.

A specimen of the plate is taken and projectiles are fired at different points on the plate at different striking velocities. If a projectile penetrates the armor it is said to have defeated the armor. Our goal is to determine the relationship between the striking velocity and the probability of penetration. Due to the expensive nature of all items involved, this goal must be achieved with a minimum amount of testing. A Bayesian approach for solving this problem is presented here and illustrated using some real data.

Research Supported by
U.S. Army Research Office
Grant DAAG-29-80-C-0067
and
Naval Surface Weapons Center under
Contract N00014-77-C-0263, with the
Office of Naval Research

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Reliability and Risk Analysis

THE U.S. ARMY (BRL'S) KINETIC ENERGY PENETRATOR PROBLEM: ESTIMATING THE PROBABILITY OF RESPONSE FOR A GIVEN STIMULUS

Thomas A. Mazzuchi
Nozer D. Singpurwalla

1. STATEMENT OF THE PROBLEM

The following statement of the problem is based on our several discussions with Dr. Robert L. Launer of the Army Research Office, Research Triangle Park, North Carolina, and Dr. J. Richard Moore of the Ballistic Research Laboratory (BRL), Aberdeen Proving Ground, Maryland.

The crew compartment of an army vehicle is protected by a certain kind of material which we will refer to as an "armor plate." It is desired to test the strength of this armor plate so that we may be able to assess its appropriateness for use on the vehicle.

In order to do this, a 10' × 10' specimen of the armor plate is taken, and a projectile is fired from a gun which is aimed at different points on the plate. In Figure 1.1 below, we indicate a possible firing pattern according to which the gun is aimed.

Typically, the distance between the muzzle of the gun and the target is about 200 meters, and the velocity of the projectile, measured

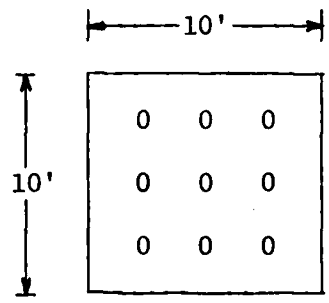

Figure 1.1--Illustration of a firing pattern of a gun.

between two conveniently located points between the gun and the target is about 5000 feet per second.

The projectile is known as the "penetrator," and the outcome of each firing is described by a binary variable which takes the value 1 if the penetrator defeats the target, and the value 0 if the penetrator fails to defeat the target. The penetrator induces a stress on the armor; the stress is a function of two quantities, the "striking velocity" and the "angle of fire." The striking velocity, also known as the "stimulus," is the velocity with which the penetrator strikes the armor, whereas the angle of fire θ (indicated in Figure 1.2 below) is the amount by which the armor plate is tilted.

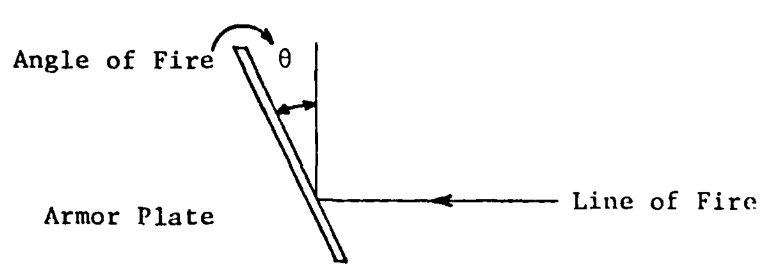

Figure 1.2--Illustration of the angle of fire.

Both the armor specimen and the penetrator are very expensive and thus the testing has to be kept to a bare minimum. One strategy that has been adopted is to fix the angle of fire, say at θ^*, and then to fire the penetrator at different striking velocities. After each firing, a record is made of whether the penetrator defeated the target or not. It is assumed that the striking velocity can be measured without any error.

2. GOALS, OBJECTIVES, AND SOME COMMENTS ON CURRENT APPROACHES

Given that our goal is to be able to assess the appropriateness of the armor plate for use on a vehicle, our objective should be to estimate the relationship between the striking velocity (the stimulus) and the probability of penetration (a response of 1). This is illustrated in Figure 2.1, wherein it is assumed that the probability of penetration is a nondecreasing function of the stimulus.

The situation described above is identical to the one encountered in "bioassay experiments," and "low dose radiation experiments," in which the relationship mentioned before is known as the quantal response curve. The dose level of a drug is the stimulus, and interest generally centers around $V_{.5}$, the stimulus at which the probability of response is .5 . Since it is possible to subject more than one animal to a particular dose level, the number of tests at each value of the stimulus can be more than one. Furthermore, tests are often conducted at several dose levels, and thus the large sample theory which typically justifies inference from bioassay experiments is adequately substantiated.

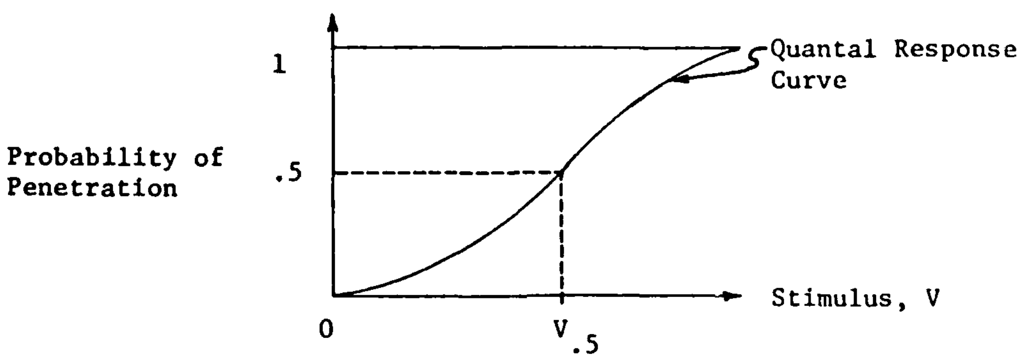

Figure 2.1--Probability of penetration vs. stimulus.

Despite these conspicuous differences between bioassay experimentation and the problem described here, the methodology and techniques of the former have been directly adopted for use in the latter. In so doing, a serious compromise has been made--the estimation of $V_{.5}$, rather than the entire quantal response curve, has been made the dominant issue of the kinetic energy penetration problem. Specifically, the BRL's commonly used "Langley Method" [Rothman, Alexander, and Zimmerman (1965, pp. 55-58)] and the "Up and Down Method" [op. cit., pp. 101-103] focus exclusive attention on the estimation of $V_{.5}$.

The typical approach used in bioassay for estimating $V_{.5}$ is to assume that the probability of response p is an arbitrary nondecreasing function of the stimulus V, specified via the relationship

$$p = F((v-\mu)/\sigma) ,$$

where F is a distribution function determined by a symmetrical density function with location parameter μ and scale parameter σ. Often F is taken to be the normal distribution function

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds ,$$

or the logistic distribution function $F(x) = (1 - e^{-x})^{-1}$.

The data from a bioassay experiment consists of n_i, the number of subjects receiving stimulus V_i, $i=1,\ldots,K$, and X_{ij}, $j=1,\ldots,n_i$, where

$$X_{ij} = 1 , \text{ if the j subject responds under stimulus } V_i \text{, and}$$
$$= 0 , \text{ otherwise.}$$

Given the data (n_i, X_{ij}), $i=1,\ldots,K$, $j=1,\ldots,n_i$, the parameters μ and σ are estimated using the method of maximum likelihood,

under the assumption that the test results can be judged independent. Once μ and σ are estimated, the estimation of $V_{.5}$ follows from the fact that F, the tolerance distribution, has been specified. Nonparametric and robust estimators of $V_{.5}$, such as the Spearman-Karber estimator, the L-estimator, the M-estimator, and the Tukey Biweight estimator, have also been obtained, all under the assumption that the density function giving F is symmetric. These estimators have been discussed by Miller and Halpern (1979). Furthermore, it has been empirically shown that for the estimation of $V_{.5}$ it does not matter what specific form is chosen for F; many of the commonly used nonparametric estimators yield identical estimates of $V_{.5}$, as long as symmetry is assumed.

A drawback of the assumption of symmetry is that the estimate of the probability of response when the stimulus is zero is nonzero. Whereas this may not be too disturbing in bioassay with its emphasis on $V_{.5}$, in the problem considered here and the low dose radiation experimentation, such an estimate would be clearly unacceptable. A zero value of the stimulus should correspond to a zero value for the probability of response.

In view of the above difficulty, the paucity of data at each level of the stimulus, and our inability to specify a functional form of F which has some practical merit, we are motivated to advocate a Bayesian approach for the solution of this problem. Our approach is described in Section 3.

3. AN OUTLINE OF A BAYESIAN APPROACH

A Bayesian approach to the bioassay problem was first proposed by Kraft and Van Eeden in 1964, and was more fully developed by Ramsey in 1972. We consider here the theme proposed by Ramsey; extensions of this theme are considered by Shaked and Singpurwalla (1982).

Let $0 \equiv V_0 < V_1 < \ldots < V_M < V_{M+1} \leqq \infty$, be M distinct levels of the stimulus at which the target (armor plate) is tested; M is chosen in advance. The outcome of a test at V_i is described by a binary (0,1) variable X_i , where $X_i = 1$ if the penetrator with a striking velocity V_i defeats the target. Let $p_i = P\{X_i=1\}$, $i=1,\ldots,M$, and without loss of generality, we assume that

$$0 \equiv p_0 < p_1 < p_2 < \ldots < p_M < p_{M+1} \equiv 1 ; \tag{3.1}$$

it is always possible to choose V_1 and V_M which satisfy the above inequality.

Given $\underset{\sim}{X} = (X_1,\ldots,X_M)$, one goal is to estimate the unknown p_i's , $i=1,\ldots,M$, subject to the inequalities (3.1). Another goal is to estimate p_j , for some $j \neq i$, such that if $V_i < V_j < V_{i+1}$, the estimates satisfy $p_i < p_j < p_{i+1}$, $i=1,\ldots,M$; this pertains to estimating the probability of response at a stimulus where no target was tested. Yet a third goal would be to estimate the largest stimulus, say V_α , for which $p_\alpha \leqq \alpha$, where $0 < \alpha < 1$ is specified.

Ramsey's approach for achieving the above goals is to assign a Dirichlet as a prior distribution for the successive differences $p_1, p_2-p_1, \ldots, p_M-p_{M-1}$, and then to use the <u>modal</u> value of the

resulting joint posterior distribution as a Bayes point estimate of $(p_1,\ldots,p_M)$. The modal value is computed with the inequalities (3.1) being satisfied. The modal value of the posterior distribution, if unique, is also known as the generalized maximum likelihood estimator [see DeGroot (1970, p. 236)], and is used as a Bayes estimator when we do not wish to specify a particular loss function. Having estimated the p_i's, the estimation of p_j and V_α is undertaken via an interpolation procedure.

Specifically, if $\alpha_i > 0$, $i=1,\ldots,M$, and $\beta > 0$ are constants such that $\sum_{i=1}^{M+1} \alpha_i = 1$, then the prior density function π is of [illegible] form

$$\pi \propto \left\{ \prod_{i=1}^{M+1} (p_i - p_{i-1})^{\alpha_i} \right\}^{\beta} . \tag{3.2}$$

It is important to note that when averaging according to π integration must be done with respect to $\prod_{i=1}^{M} dp_i \,/\, \prod_{i=1}^{M+1} (p_i - p_{i-1})$.

Since M has been prechosen, the stopping rule is clearly delineated, and so the likelihood for the response probabilities at the observed stresses is

$$\prod_{i=1}^{M} p_i^{X_i} (1 - p_i)^{1-X_i} . \tag{3.3}$$

The joint density function of the posterior distribution of $p_1,\ldots,p_M$ is proportional to the product of the prior density function (3.2) and the likelihood function (3.3). Thus

$$f(p_1,\ldots,p_M \mid X_1,\ldots,X_M)$$

$$\propto \prod_{i=1}^{M+1} p_i^{X_i}(1-p_i)^{1-X_i}\left[\frac{\Gamma(\beta)}{\prod_{i=1}^{M+1}\Gamma(\beta\alpha_i)}\right]\left\{\prod_{i=1}^{M+1}(p_i-p_{i-1})^{\alpha_i}\right\}^{\beta} . \qquad (3.4)$$

Ramsey has not been able to obtain the posterior marginal distributions of p_i, $i=1,\ldots,M$, nor has he commented on any aspects of these distributions. He uses a nonlinear programming algorithm to obtain $(\hat{p}_1,\ldots,\hat{p}_M)$, the modal value of (3.4), subject to the constraint that $\hat{p}_1 \leq \hat{p}_2 \leq \cdots \leq \hat{p}_M$; this is his Bayes estimator of $(p_1,\ldots,p_M)$. In contrast to this Mazzuchi (1982) has been able to obtain all the moments of the marginal posterior distribution of the p_i, $i=1,\ldots,M$. This work of Mazzuchi's represents an extension of Ramsey's results, and is one that takes us a step closer to a fully Bayesian analysis. The moments can be used to approximate the marginal posterior distributions of the p_i's using the techniques given in Elderton and Johnson (1969). The approximated posterior distributions give us a measure of uncertainty associated with our using the first moment of the marginal posterior distribution of p_i, $i=1,\ldots,M$, as our Bayes estimate of p_i. The first moment of the marginal posterior distribution is used as a Bayes estimator when we are willing to assume the square error as a loss function. The formulae for the moments and their use for approximating the marginal posterior distributions are given in Appendix A.

The computational effort required to compute the moments mentioned above increases with M. Thus there is a trade-off between the convenience of using an optimization algorithm to obtain the modal value

of (3.4), versus the laborious computational effort involved in obtaining several moments of each of the M posterior marginal distributions. The optimization algorithm cited above is based on the "Sequential Unconstrained Minimization Technique" (SUMT) of Fiacco and McCormick (1968). A computer code which adopts SUMT for the problem considered here is described by Mazzuchi and Soyer (1982). This code can also be used for the computation of the moments of the marginal posterior distributions of the p_i , $i=1,\ldots,M$.

3.1 Specification of the Prior Parameters

In order to implement the Bayesian procedure, we need to specify the prior parameters α_i , $i=1,\ldots,M$, and β , given in (3.2). In order to do this, we observe (see Ramsey) that $u_i = p_i - p_{i-1}$, $i=1,\ldots,M$, has a beta distribution on the unit interval (denoted as $u_i \sim \text{Beta}(\beta\alpha_i, \beta(1-\alpha_i); 0,1))$,

$$f(u_i;\ \beta\alpha_i,\ \beta(1-\alpha_i)) = \frac{\Gamma(\beta)}{\Gamma(\beta\alpha_i)\ \Gamma(\beta(1-\alpha_i))}\, u_i^{\beta\alpha_i}\ (1-u_i)^{\beta(1-\alpha_i)}\ , \quad 0 \leq u_i \leq 1\ ,$$

with

$$E(u_i) = \alpha_i\ , \text{ and} \tag{3.5}$$

$$\text{Var}(u_i) = \frac{\alpha_i(1-\alpha_i)}{(\beta+1)}\ . \tag{3.6}$$

If P_i^* denotes our best prior guess about P_i , consistent with the fact that the P_i^*'s increase in i , then the α_i's can be obtained via (3.5) as

$$\alpha_1 = P_1^*$$

$$\alpha_i = P_i^* - P_{i-1}^*, \quad i=2,\dots,M,$$

and

$$\alpha_{M+1} = 1 - P_M^* .$$

In order to choose the parameter β, we need to have some idea about the uncertainty associated with our choice of p_1^*. This in practice can be done in one of the following two ways:

(i) Suppose that in addition to p_1^*, our best guess about the variance of p_1 is $\mathrm{Var}(p_1)$. Then, substituting $\alpha_1 = p_1^*$ in (3.6), we have

$$\mathrm{Var}(u_1) = \mathrm{Var}(p_1) = \frac{\alpha_1(1-\alpha_1)}{(\beta+1)},$$

so that

$$\beta = \begin{cases} \dfrac{p_1^*(1-p_1^*)}{\mathrm{Var}(p_1)} - 1, & \text{if } \beta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\beta = 0$ corresponds to the case of isotonic regression.

(ii) Often in practice [cf. McDonald (1979)], associated with the best guess value p_1^*, a user is able to specify two numbers $a_1^* > 0$ and $b_1^* < 1$, such that for some γ_1 (specified by the user), $0 < \gamma_1 < 1$,

$$P(a_1^* < p_1 < b_1^*) = 1 - \gamma_1 .$$

Since $p_1 \sim \mathrm{Beta}(\beta\alpha_i, \beta(1-\alpha_i); 0,1)$, given p_1^*, we set $\alpha_1 = p_1^*$, and find that value of β such that

$$\int_{a_1^*}^{b_1^*} \frac{\Gamma(\beta)}{\Gamma(\beta\alpha_1)\,\Gamma(\beta(1-\alpha_1))}\, p_1^{\beta\alpha_1-1} (1-p_1)^{\beta(1-\alpha_1)-1}\, dp_1 = 1-\gamma_1 \tag{3.7}$$

Suppose, further, that for any one or more of the indices i , $i=2,\ldots,M$, a user is also able to specify two numbers $a_i^* > p_{i-1}^*$, and $b_i^* < 1$, such that for some γ_i (specified by the user), $0 < \gamma_i < 1$,

$$P(a_i^* < (p_i^* \mid p_{i-1}^*,\ldots,p_1^*) < b_i^*) \;=\; 1 - \gamma_i \; .$$

Then, using the fact (see Ramsey) that

$$(p_i \mid p_{i-1}^*) \sim \mathrm{Beta}(\beta\alpha_i,\ \beta(1 - \alpha_1 - \ldots - \alpha_i);\ p_{i-1}^*,\ 1)$$

$$= f(p_i \mid p_{i-1}^*;\ \beta,\alpha_i) \text{ , say,}$$

we can find the <u>smallest</u> value of β , β^* , which satisfies (3.7) and (3.8), where

$$\int_{a_i^*}^{b_i^*} f(p_i \mid p_{i-1}^*;\ \beta,\alpha_i)dp_i \;=\; 1 - \gamma_i \text{ ,} \tag{3.8}$$

with $\alpha_i \;=\; p_i^* - p_{i-1}^*$, $i=2,\ldots,M$.

A computer code which determines the smallest value of β described above is available; the details of this program are given by Mazzuchi and Soyer (1982). Our reason for choosing the smallest value of β stems from the fact that large values of β give a very strong prior, with the result that even a large amount of failure data will not change our prior distribution.

3.2 Interpolation Procedure and the Estimation of Quantiles

Let the M-dimensional point

$$(p_1^+,\ldots,p_M^+) = \begin{cases} (\hat{p}_1,\ldots,\hat{p}_M) & \text{, if the mode of the joint posterior is} \\ (\tilde{p}_1,\ldots,\tilde{p}_M) & \text{, if the first moments of the marginal posterior are} \end{cases}$$

used as the Bayes estimator of $(p_1,\ldots,p_M)$.

Suppose that we wish to estimate p_j , for some $j \neq i$, $i=1,\ldots,M$, where $V_i < V_j < V_{i+1}$. Let p_j^* be our best prior guess of p_j , the probability of response at a nonexperimental impulse V_j . Then, following Ramsey, we pick p_j^+ in such a manner that

$$\frac{p_{i+1}^* - p_j^*}{p_{i+1}^+ - p_j^+} = \frac{p_j^* - p_i^*}{p_j^+ - p_i^+} . \tag{3.9}$$

For the estimation of V_α , the αth quantile $(0 < \alpha < 1)$, we first see if there is an observation stimulus, say V_i , for which $p_i^+ = \alpha$. If so, then V_i is our Bayes estimate of V_α . If not, we determine the pair of observational impulses, say V_i and V_{i+1} , for which $p_i^+ < \alpha < p_{i+1}^+$. Since the probability of response curve is assumed to be increasing, the straight line segment joining the points $0, p_1^+, \ldots, p_i^+, p_{i+1}^+, \ldots, p_M^+, 1$, will be an increasing function of i . We shall find that value of the impulse, say V_α^+ , $V_i < V^+ < V_{i+1}$, for which $p_\alpha^+ = \alpha$.

4. APPLICATION TO SOME BRL DATA

In Appendix B we present eight sets of data labelled 1, 2, 3, 4, 6, 7, 8, and 9, pertaining to 60 kinetic energy penetration tests. These data were given to us by Dr. Moore of BRL and have been carefully sanitized to maintain confidentiality. Data sets labelled 5 and 10, also given to us by Dr. Moore, have been eliminated from consideration because the striking velocity for these data is much too different from those of the other sets. All the 10 sets of data were obtained sequentially over time, in the sense that data set 1 was the first one to be obtained, followed by data set 2 (obtained after some lapse of time), and so on, until we reach data set 9, which is the last considered here. To the best of our knowledge, all eight data sets are assumed to have been collected under identical conditions. That is, there is no indication that, except for differences in striking velocity, the material and the methods of testing used for data set 1 are different from those used in data set 2, and so on. This, plus the sequential nature of the data, enables us to use the posterior obtained from one data set as the prior for the next set, and so on, until we obtain the posterior using data set 9, which then gives our final estimate of the response curve.

Data set 1 consists of 13 observations taken at striking velocities ranging from 128.60 (in some unspecified units) to 166.16. The result of each test is indicated by a binary variable X_i. The best prior guess values p_i^*, necessary to choose the prior parameters α_i, were not specified by BRL. However, what appears to be reasonable is to assume that the probability of response at a striking velocity of 100 is close to zero, and that at a striking velocity of 200 it is almost 1.

Thus we make an arbitrary choice for p_i^*, say p_{i0}^*, by letting $p_{i0}^* = 1 - \exp[-.07(V_i - 100)]$. Data on striking velocities outside the range of 100 to 200 were excluded. Despite this arbitrary choice of p_{i0}^*, we shall see how even a scant amount of data significantly changes the posterior response curve, provided that the smoothing parameter β is not too large. Three values of β were also chosen arbitrarily; these are 1, 10, and 25. Recall that small values of β tend to emphasize the data, whereas large values of β tend to emphasize the prior distribution. In Appendix B we show our analysis for the case of $\beta = 10$.

Since, in reality, the data are generated sequentially over time, our first step would be to revise the best prior guess values p_{i0}^*, $i=1,\ldots,61$, based on data set 1 alone. The posterior (modal) values corresponding to the striking velocities of data set 1, p_{i1}^+, will be the revised values of p_{i0}^*, for $i=1,\ldots,13$; these are given in column 5 of the table in Appendix B. The revised values of p_{i0}^*, for $i=4,\ldots,61$, are obtained via the interpolation formula (3.9), using p_{i1}^+, $i=1,\ldots,13$, and p_{i0}^*, $i=14,\ldots,61$. Let the revised values of p_{i0}^*, $i=14,\ldots,61$, be denoted by p_{i1}^*; these too are shown in column 5 of the table in Appendix B.

Upon receiving data set 2, we revise the values p_{i1}^*, $i=14,\ldots,19$, by the posterior modal values corresponding to the six striking velocities of data set 2. We denote these revised values by p_{i2}^+, $i=14,\ldots,19$; these are given in column 5 of the table in Appendix B. The revised values of p_{i1}^*, $i=20,\ldots,61$, are obtained by interpolation, using

p^+_{i1} , i=1,...,13 , p^+_{i2} , i=14,...,19 , and p^*_{i1} , i=20,...,61 ; we denote these revised values by p^*_{i2} , i=20,...,61 , and show them in column 6.

We continue the above scheme of systematically revising the $p_{i.}$'s , either via the posterior modal values or by interpolation, until we incorporate the effect of all eight sets of data. Data set 9, the last one considered here, consists of eight observations taken at starting velocities ranging from $V_{54} = 144.83$ to $V_{61} = 198.94$. The posterior modal values corresponding to the striking velocities of data set 9, p^+_{i8} , i=54,...,61 , are given in column 12; the interpolated values p^*_{i7} required to obtain the p^+_{i8}'s are given in column 11. Since the p^*_{i7}'s incorporate the results of the previous seven sets of data, we claim that the final posterior modal values p^+_{i8} , i=54,...,61 , are based on the results of all the testing. Had we ignored the sequential nature of the data and computed the posterior modal values by using Bayes Theorem on the best prior guess values p^*_{i0} , i=1,...,61 , then the posterior modal values corresponding to V_{14} through V_{61} would be different from the $p^+_{i.}$ values, i=14,...,61 , given in the table. This difference is due to the interpolation scheme that is used to constantly revise the best prior guess values, when we consider the data sets sequentially.

A plot of p^+_{i8} versus V_i , i=54,...,61 , represents our final estimate of the quantal response curve. Estimates of the probabilities of response at striking velocities different from V_i , i=54,...,61 , can be obtained using the interpolation formula (3.9). When we use the interpolation formula to obtain an estimate of p_j , for some

$j=1,\ldots,53$, we need to specify a value p_j^* , the best prior guess value of p_j . Suppose that the index j appears in data set k , for some $k < 9$; then for p_j^* we will use p_{jk}^+ . In so doing, we will have incorporated the effect of the last data set, data set 9, in our obtaining the estimate of p_j , and thus achieve a certain amount of smoothness. Note that the effect of the data sets between k and 9 is already present in our estimates p_{i8}^+ , $i=54,\ldots,61$, and these are used in our interpolation scheme. For example, suppose that we wish to estimate the probability of response at a striking velocity of 158.52. This striking velocity occurs in data set 2, and lies between the striking velocities 148.97 and 159.15 of data set 9. The index j corresponding to the value 158.82 is 17. To use (3.9), we identify p_{i+1}^* and p_{i+1}^+ as being .70499 and .53014, respectively, p_i^* and p_i^+ as .62881 and .42386 (see data set 9), and p_j^* as .64436 (see data set 2), and compute p_j^+ as our estimate of p_j .

In Figures 4.1, 4.2, and 4.3, we show plots of our Bayes estimate of the probability of response at the eight striking velocities of data set 9, for $\beta = 1, 10,$ and 25 , respectively. Also shown are the 90% probability of coverage intervals for each estimate. These intervals are obtained using the moments of the posterior distributions of p_i , $i=54,\ldots,61$, and then using the techniques of Elderton and Johnson (1969) to approximate the posterior distributions--see Appendix A. On each of these figures we also show a graph of our best guess values p_{i0}^* , $i=1,\ldots,61$; these enable us to see how the data have changed our prior estimates. We observe that the 90% probability of coverage intervals tend to be small in the middle of the range of the striking velocities.

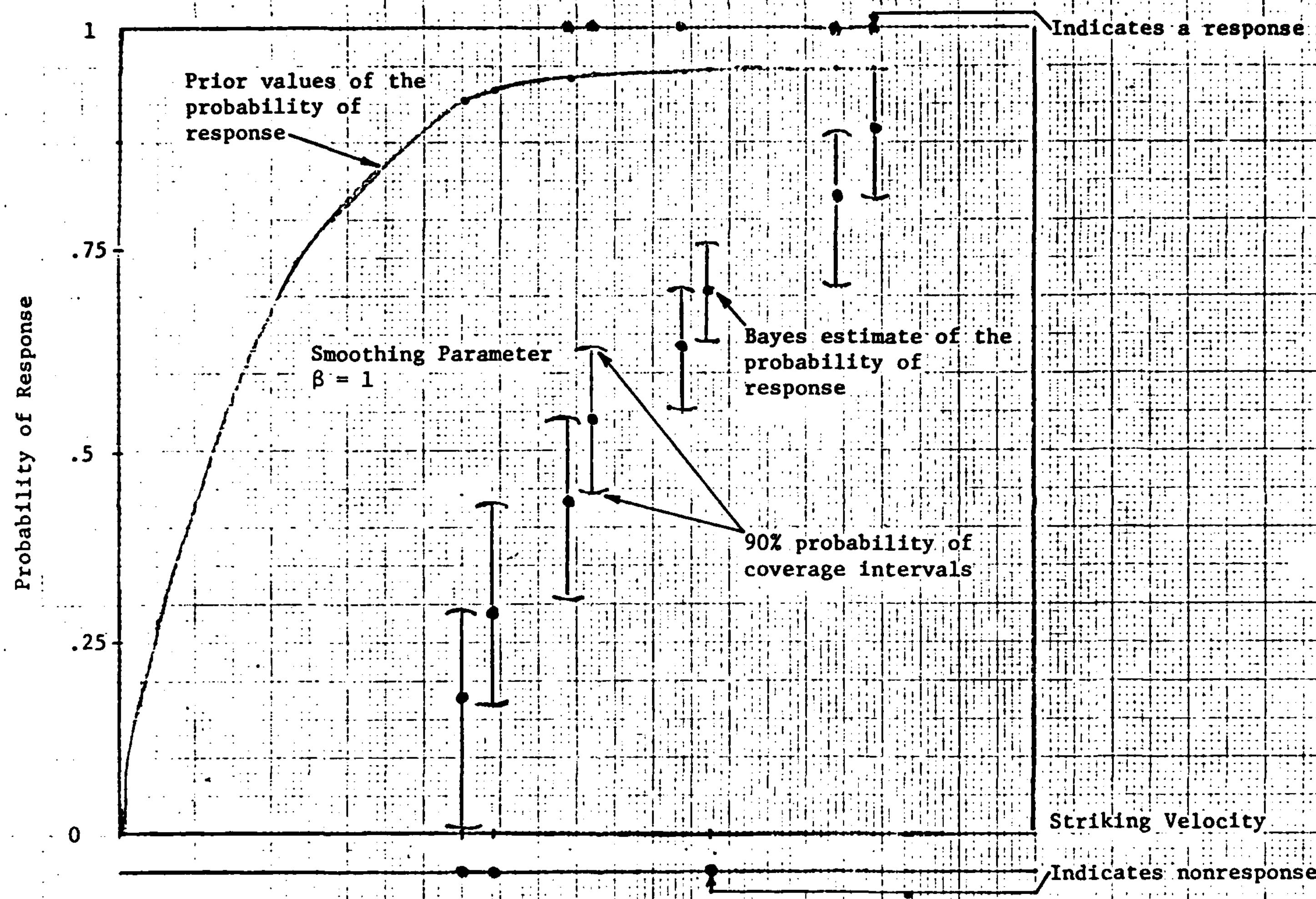

Figure 4.1--Prior and posterior values of the probability of response for data set 9, with $\beta = 1$.

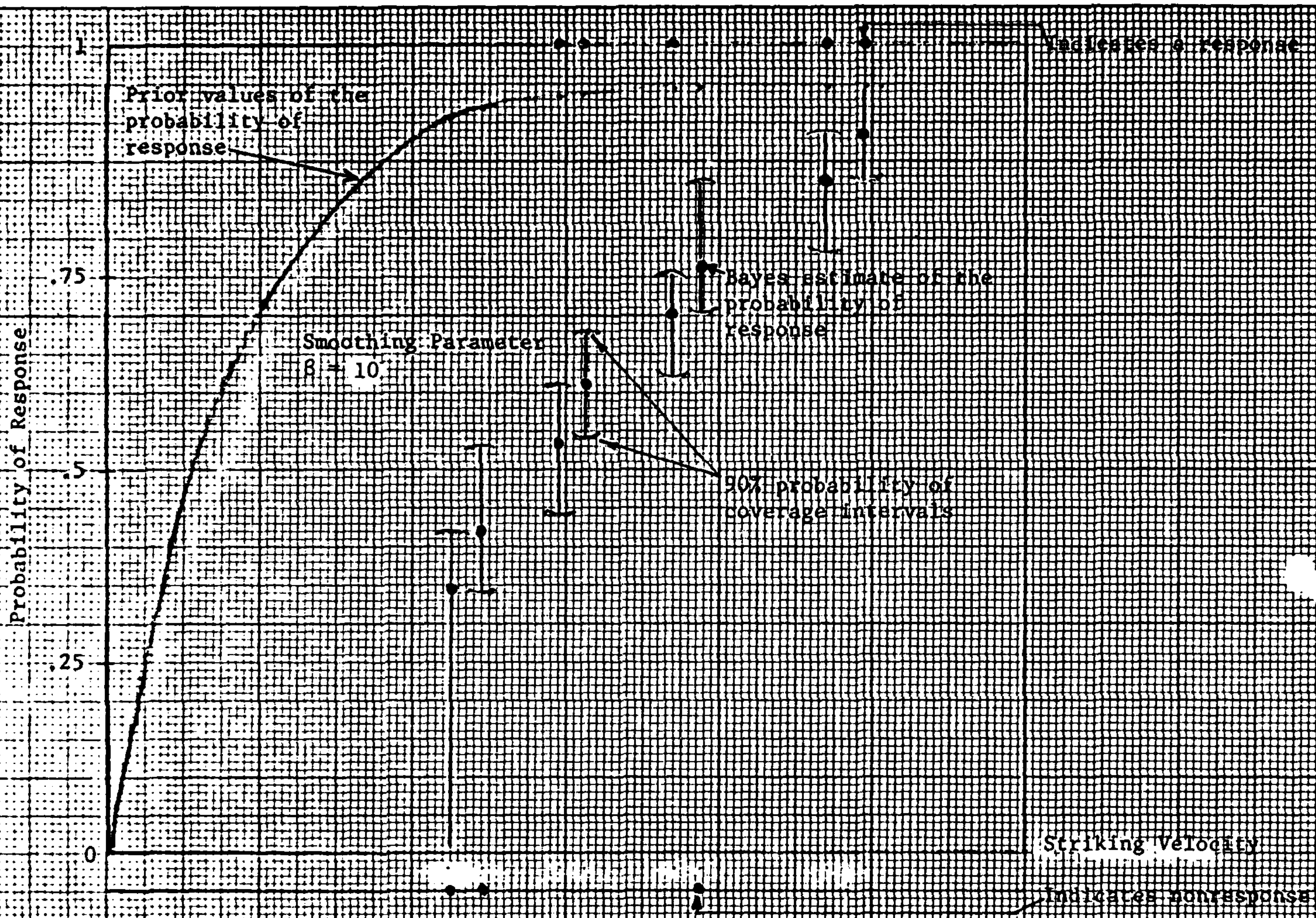

Figure 4.2—Prior and posterior values of the probability of response for data set 9, with β = 10

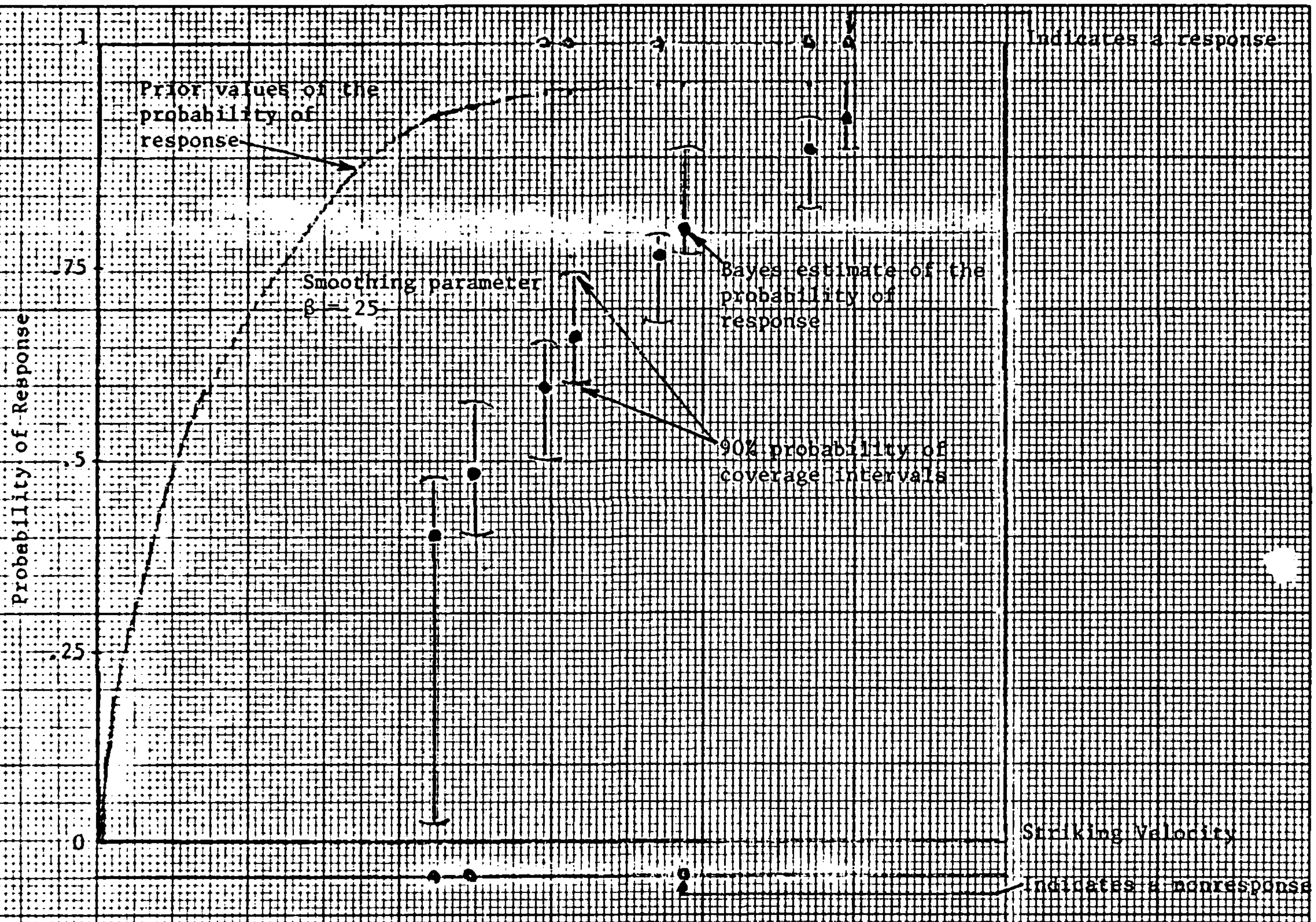

Figure 4.3—Prior and posterior values of the probability of response for data set 9, with β = 25.

In Figure 4.4, we superimpose the plots of Figures 4.1, 4.2, and 4.3, in order to give a perspective of the effect of β in our computations. It appears that our Bayes estimates for the three cases of $\beta = 1, 10,$ and 25 tend to converge toward each other; this is to be expected, since we have 61 observations with which we revise our prior probabilities.

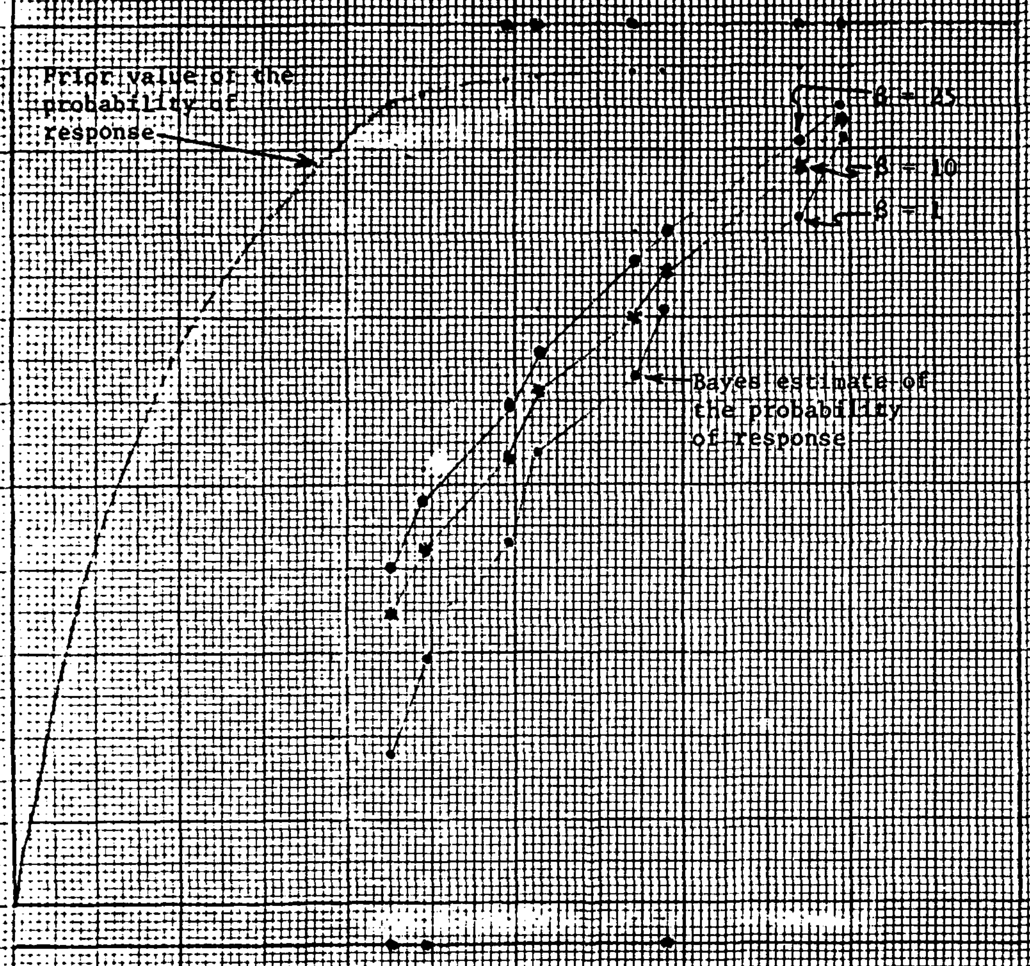

Figure 4.4--Prior and posterior values of the probability of response for β = 1, 10, and 25.

ACKNOWLEDGMENTS

The use of the "Sequential Unconstrained Minimization Technique" (SUMT) was central to our undertaking and accomplishing the work reported here; without it, we would not have been able to implement our Bayesian ideas. We gratefully acknowledge the help of Professors A. V. Fiacco and G. P. McCormick in connection with our use of SUMT. We also acknowledge the several helpful conversations with Drs. J. Richard Moore and Robert Launer regarding several aspects of this problem, and are particularly grateful to the latter for encouraging us to write this up. Mr. Refik Soyer's assistance with developing the computer programs cited here is also recognized. Mr. W. McDonald of the Naval Surface Weapons Center, made several comments which led us to develop the material of Section 3.1.

APPENDIX A

Moments of the Marginal Posterior Distributions

The moments of the posterior distribution of p_i , $i=1,\dots,M$, have been obtained by Mazzuchi (1982); a formula for obtaining these is given below. A computer code which facilitates the computation of the moments is described by Mazzuchi and Soyer (1982).

Let $\bar{X}_i = 1 - X_i$, $i=1,\dots,M$, $B(a,b) = \Gamma(a)\Gamma(b) / \Gamma(a+b)$, and

$$K = \sum_{r_1=0}^{\bar{X}_1} \cdots \sum_{r_M=0}^{\bar{X}_M} (-1)^{\sum_{i=1}^{M} r_i} \prod_{i=1}^{M} B\left(\sum_{j=1}^{i} X_j + \beta\alpha_j + r_j,\ \beta\alpha_{i+1} \right) .$$

Then, for $\ell=1,2,\dots$,

$$E(p_s^{\ell}) = \frac{1}{K} \sum_{r_1=0}^{\bar{X}_1} \cdots \sum_{r_M=0}^{\bar{X}_M} (-1)^{\sum_{i=1}^{M} r_i} \prod_{i=1}^{M} B\left(\sum_{j=1}^{i} X_j^* + \beta\alpha_j + r_j,\ \beta\alpha_{i+1} \right) ,$$

where

$$X_j^* = \begin{cases} X_j + \ell , & j = s \\ X_j , & \text{otherwise.} \end{cases}$$

These moments can be used to approximate the posterior distribution of p_i , $f(p_i)$, $i=1,\dots,M$. In order to do this, we consider a system of frequency curves described by Elderton and Johnson (1969) which are based on the transforms of a standard normal variate Z . The system of curves which is appropriate to our problem is that referred to as the "bounded system of curves," denoted by Elderton and Johnson (1969, p. 123) as S_B , and described by

$$Z = \gamma + \delta \, \mathrm{Ln}[(p_i - \varepsilon) / (\varepsilon + \lambda - p_i)] , \quad \varepsilon < p_i < \varepsilon+\lambda ,$$

where γ , δ , λ , and ε are parameters whose values are determined by the first four moments of $f(p_i)$ about its mean.

Hill, Hill, and Holder (1976) give a computer code which determines γ , δ , λ , and ε from the first four moments of $f(p_i)$ about its mean. Since it was assumed that $p_{i-1} < p_i < p_{i+1}$, we estimate λ and ε from the Bayesian estimates of the p_i ; γ and δ are obtained from the computer code. Having obtained these parameters, the distribution $f(p_i)$ is obtained from Elderton and Johnson (1969, p. 130) as

$$f(p_i) = \frac{N}{\lambda\sqrt{2\pi}} \left[\left(\frac{p_i-\varepsilon}{\lambda}\right)\left(1 - \frac{p_i-\varepsilon}{\lambda}\right)\right]^{-1} \exp\left[-\tfrac{1}{2}\left(\gamma + \delta\ell n\left(\frac{p_i-\varepsilon}{\varepsilon+\lambda-p_i}\right)\right)^2\right],$$

$$\varepsilon < p_i < \varepsilon + \lambda ,$$

where N in our case is 1.

In order to obtain the approximate $(1-\gamma)\%$ probability of coverage intervals for each p_i , which contain its Bayes estimate $\overset{+}{p}_i$ (mode or mean), we use the fact that since

$$z = \gamma + \delta\ell n[(p_i-\varepsilon)/(\varepsilon+\lambda-p_i)] , \quad \varepsilon < p_i < \varepsilon + \lambda ,$$

$$p_i = \lambda \exp\left[\left(\frac{\gamma-z}{\delta}\right) + 1\right]^{-1} + \varepsilon .$$

Thus, to find two numbers, a and b , such that

$$P\{\overset{+}{p}_i-a \leqq p_i \leqq \overset{+}{p}_i+b\} = 1 - \delta ,$$

we use

$$P\left\{-\delta\ell n\left(\frac{\lambda}{\overset{+}{p}_i-a-\varepsilon} - 1\right) + \gamma \leqq z \leqq -\delta\ell n\left(\frac{\lambda}{\overset{+}{p}_i+b-\varepsilon} - 1\right) + \gamma\right\} = 1 - \delta ,$$

and solve for a and b by setting

$$-\delta \ell n \left(\frac{\lambda}{\overset{+}{p}_i - a - \varepsilon} - 1 \right) + \gamma = z_{1-(\delta/2)}$$

and

$$-\delta \ell n \left(\frac{\lambda}{\overset{+}{p}_i + b - \varepsilon} - 1 \right) + \gamma = z_{\delta/2} ,$$

where $z_{\delta/2}$ is the (1-(δ/2))th percentile of a standard normal distribution. Taking $c = \max(a,b)$, we form our interval

$$\Pr\{\overset{+}{p}_i - c < p_i < \overset{+}{p}_i + c\} \geqq 1 - \delta .$$

These intervals may not be symmetric about the mean or modal estimate. This case arises when the boundaries of the probability of coverage interval exceed the boundary of the variable. In such cases the variable boundary is used as the boundary of the probability of coverage interval. The probability of any symmetric interval about the mean or modal estimate may be obtained by proceeding in the reverse or the above and evaluating the interval for the standard normal variate.

APPENDIX B

In the table below we give values of the striking velocity V_i , the response X_i , and the best prior guess values p^*_{i0} , $i=1,\ldots,61$, for the eight sets of data described in Section 4. We also show, for $\beta = 10$, the revised values of p^*_{i0} , p^+_{ij} , or p^*_{ij} based on data set j , $j=1,2,3,4,6,7,8,9$.

TABLE B.1

Data Set	Striking Velocity V_i	Response X_i	Best Guess Values P^*_{i0}	Revised Values P^+_{i1}, i=1,...,13 and P^*_{i1}, i=14,...,61	Revised Values P^+_{i2}, i=14,...,19 and P^*_{i2}, i=20,...,61	Revised Values P^+_{i3}, i=20,...,26 and P^*_{i3}, i=27,...,61	Revised Values P^+_{i4}, i=27,...,33 and P^*_{i4}, i=34,...,61	Revised Values P^+_{i6}, i=34,...,39 and P^*_{i6}, i=40,...,61	Revised Values P^+_{i7}, i=40,...,45 and P^*_{i7}, i=46,...,61	Revised Values P^+_{i8}, i=46,...,53 and P^*_{i8}, i=54,...,61	Revised Values P^+_{i9}, i=54,...,61	Observation No.
1	128.60	0	.86141	.29717								1
	132.74	0	.89589	.34071								2
	143.56	0	.95071	.39334								3
	146.74	0	.96043	.43289								4
	147.70	1	.96297	.47221								5
	149.29	0	.96682	.50928								6
	153.11	0	.97452	.55059								7
	154.70	0	.97717	.59361								8
	155.34	0	.97816	.64076								9
	159.15	0	.98321	.69702								10
	161.70	1	.98593	.76390								11
	162.79	1	.98695	.82463								12
	166.16	0	.98966	.88210								13
2	139.42	0	.93438	.37766	.25028							14
	145.15	0	.95584	.41421	.32728							15
	145.79	0	.95775	.42199	.39349							16
	158.52	1	.98245	.68855	.64436							17
	163.61	1	.98767	.83990	.79980							18
	169.98	1	.99206	.90947	.89721							19
3	110.14	0	.50375	.17378	.11517	.13815						20
	120.28	1	.85831	.29610	.19623	.26379						21
	130.51	0	.87855	.31881	.21128	.32700						22
	133.05	1	.89810	.34283	.22720	.39629						23
	146.24	1	.95904	.42723	.39842	.53722						24
	148.01	0	.96376	.47982	.44792	.60804						25
	161.06	1	.98529	.74816	.70558	.80077						26
4	118.09	0	.71350	.24614	.16312	.21247	.16467					27
	132.10	0	.89112	.33478	.22186	.37305	.31127					28
	137.83	0	.92676	.37035	.24544	.41130	.39592					29
	141.97	1	.94498	.38784	.27173	.43294	.47888					30
	148.01	1	.96376	.47982	.44792	.60804	.64292					31
	155.65	1	.97892	.64589	.60421	.72495	.76144					32
	183.03	1	.99678	.96328	.95831	.97179	.93809					33
6	120.32	0	.75442	.26026	.17248	.22698	.17792	.15360				34
	130.83	0	.88120	.32216	.21350	.33666	.27805	.27191				35
	138.15	0	.92836	.37188	.24645	.41213	.39910	.41406				36
	155.65	1	.97862	.64589	.60421	.72495	.76144	.74847				37
	155.65	1	.97862	.64589	.60421	.72495	.76144	.74847				38
	156.29	1	.97955	.65265	.61396	.73224	.76666	.81215				39
7	158.52	0	.98247	.68878	.64458	.75514	.78805	.82534	.48173			40
	159.42	0	.98353	.70489	.66114	.76753	.79191	.83248	.54375			41
	169.34	1	.99170	.90536	.89146	.92655	.90571	.92409	.67075			42
	173.45	0	.99376	.92885	.91921	.94533	.91915	.93491	.71758			43
	185.26	1	.99724	.96853	.96427	.97582	.95093	.95728	.83340			44

Table B.1—*Continued*

Data Set	Striking Velocity V_i	Response X_i	Best Guess Values P^*_{i0}	Revised Values P^+_{i1}, i=1,...,13 and P^*_{i1}, i=14,...,61	Revised Values P^+_{i2}, i=14,...,19 and P^*_{i2}, i=20,...,61	Revised Values P^+_{i3}, i=20,...,26 and P^*_{i3}, i=27,...,61	Revised Values P^+_{i4}, i=27,...,33 and P^*_{i4}, i=34,...,61	Revised Values P^+_{i6}, i=34,...,39 and P^*_{i6}, i=40,...,61	Revised Values P^+_{i7}, i=40,...,45 and P^*_{i7}, i=46,...,61	Revised Values P^+_{i8}, i=46,...,53 and P^*_{i8}, i=54,...,61	Revised Values P^+_{i9}, i=54,...,61	Observation No.
	187.17	1	.99758	.97241	.96867	.97880	.95347	.96254	.90878			45
8	102.18	0	.13984	.04824	.03197	.03835	.02972	.02566	.01498	.05849		46
	116.50	0	.68023	.23467	.15552	.20069	.15554	.13428	.07838	.14365		47
	119.37	0	.73775	.25451	.16867	.22107	.17252	.14894	.08693	.20861		48
	124.14	0	.81139	.27991	.18550	.24716	.19634	.17536	.10235	.27999		49
	134.01	1	.90464	.34911	.23136	.39971	.37027	.38021	.22192	.42858		50
	134.33	1	.90673	.35112	.23269	.40081	.37271	.38307	.22359	.48811		51
	141.01	1	.94121	.38422	.26410	.42666	.45480	.46547	.27168	.56532		52
	175.07	1	.99441	.93626	.92763	.95103	.92323	.93820	.75167	.84252		53
9	144.83	0	.95485	.41019	.31881	.47169	.51518	.52119	.30421	.58411	.34483	54
	148.97	0	.96608	.50216	.46894	.62376	.65886	.65380	.38161	.62881	.42386	55
	159.15	1	.98321	.69702	.65306	.76149	.78759	.82900	.51352	.70499	.53014	56
	162.34	1	.98656	.77890	.73715	.82213	.83099	.86394	.58736	.74763	.60749	57
	174.12	1	.99403	.93193	.92271	.94770	.92085	.93628	.74345	.83777	.70205	58
	177.30	0	.99521	.94538	.93798	.95803	.92824	.94223	.76893	.85347	.75580	59
	194.17	1	.99851	.98301	.98071	.98695	.97136	.97694	.94385	.96439	.87678	60
	198.94	1	.99893	.98780	.98615	.99063	.97944	.98345	.95970	.97444	.93611	61

REFERENCES

DE GROOT, M. H. (1970). *Optimal Statistical Decisions.* McGraw-Hill, New York.

ELDERTON, W. P. and N. L. JOHNSON (1969). *Systems of Frequency Curves.* Cambridge University Press, London.

FIACCO, A. V. and G. P. McCORMICK (1968). *Nonlinear Programming: Sequential Unconstrained Minimization Technique.* Wiley, New York.

HILL, L. T., R. HILL, and R. L. HOLDER (1976). Fitting Johnson curves by moments. *Appl. Statist. 25*, 180-190.

KRAFT, C. H. and C. VAN EEDEN (1964). Bayesian bioassay. *Ann. Math. Statist. 7*, 163-186.

MAZZUCHI, T. A. (1982). Bayesian nonparametric estimation of the failure rate. D.Sc. dissertation, George Washington University.

MAZZUCHI, T. A. and R. SOYER (1982). Computer programs for 'A Bayesian approach to quantile and response probability estimation using binary response data'--A user's guide. Technical Paper Serial GWU/IRRA/TR-82/5, George Washington University.

McDONALD, W. (1979). A statistical treatment of the explosion damage problem, with applications to the case of damage to submarine pressure hulls by very low yield nuclear weapons. Technical Report, Naval Surface Weapons Center, White Oak, Maryland.

MILLER, R. G. and J. W. HALPERN (1979). Robust estimators for quantile bioassay. Technical Report No. 42, Division of Biostatistics, Stanford University.

RAMSEY, L. L. (1972). A Bayesian approach to bioassay. *Biometrics 28*, 841-858.

ROTHMAN, D., M. J. ALEXANDER, and J. M. ZIMMERMAN (1965). The design and analysis of sensitivity experiments, Volume 1. Technical Report NAS8-11061, National Aeronautics and Space Administration.

SHAKED, M. and N. D. SINGPURWALLA (1982). A Bayesian approach to quantile and response probability estimation using binary response data. Technical Report in progress.